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Annulus amplitudes and ZZ branes in minimal string theory

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Abstract: We study the annulus amplitudes of (p, q) minimal string theory. Focusing on the ZZ-FZZT annulus amplitude as a target-space probe of the ZZ brane, we use it to confirm that the ZZ branes are localized in the strong-coupling region. Along the way we learn that the ZZ-FZZT open strings are fermions, even though our theory is bosonic! We also provide a geometrical interpretation of the annulus amplitudes in terms of the Riemann surface $\mathcal{M}_{p,q}$ that emerges from the FZZT branes. The ZZ-FZZT annulus amplitude measures the deformation of $\mathcal{M}_{p,q}$ due to the presence of background ZZ branes; each kind of ZZ-brane deforms only one A-period of the surface. Finally, we use the annulus amplitudes to argue that the ZZ branes can be regarded as "wrong-branch" tachyons which violate the bound $\alpha < Q/2$.

Keywords: D-branes, 2D Gravity, Matrix Models.

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Contents

1. Introduction

Minimal string theories are string theories based on the (p,q) minimal conformal field theories coupled to Liouville. They were first solved using their description in terms of matrix models [1]–[8] (for reviews, see e.g. [9, 10]). This tractable description allows explicit calculations of many physical quantities of interest. In particular, it has already offered many insights regarding nonperturbative phenomena, D-branes and holography.

Recent progress in the study of Liouville theory [11]–[13] and its D-branes [14]–[17] has motivated the renewed interest in the subject [18]–[39]. In this note we continue the investigation started in [31], whose purpose was to explore the D-branes of minimal string theory and to derive the matrix model starting from the worldsheet description. Following [19] we will focus on the annulus diagram as an interesting diagnostic of the theory. We will use it to understand the connection between the target space fields in the theory, the D-branes and their interpretation in terms of an auxiliary Riemann surface.

Before we begin, let us briefly review (p,q) minimal string theory. These theories consist of (p,q) minimal CFT coupled to Liouville theory with background charge $Q = b+b^{-1}$, and they exist for all relatively prime integers p and q. Our convention throughout will be $p<q$. The parameter b, entering also in the Liouville potential $\mu e^{2b\phi}$, should be

set to

$$b = \sqrt{\frac{p}{q}} \tag{1.1}$$

in order for the total central charge of the Liouville and the minimal model to be 26. In the following, we will set $\mu = 1$ for simplicity (after rescaling it by $\pi\gamma(b^2)$, as in [31]).

Although the minimal string theories have physical closed-string vertex operators at every ghost number less than or equal to one, in this paper we will focus our attention on the "tachyon" operators $\mathcal{T}_{r,s}$ at ghost number one and the ground ring operators $\widehat{\mathcal{O}}_{r,s}$ at ghost number zero. Both sets of operators are labelled by integers $r = 1, \ldots, p-1$ and $s = 1, \ldots, q-1$, although the tachyons obey an additional reflection relation $\mathcal{T}_{p-r,q-s} = \mathcal{T}_{r,s}$ that cuts their number in half.

One of the results of [31] was the emergence of an auxiliary Riemann surface $\mathcal{M}_{p,q}$ which underlies many, if not all, of the features of minimal string theory. The Riemann surface appeared from the study of a class of D-branes of minimal string theory called FZZT branes [14, 15]. It is described by the algebraic equation

$$F(x,y) \equiv T_q(x) - T_p(y) = 0\,, \tag{1.2}$$

where $T_p(\cos\theta) = \cos p\theta$ is the Chebyshev polynomial of the first kind. Here

$$x = \mu_B, \qquad y = \partial_{\mu_B} Z(\mu_B) \tag{1.3}$$

correspond to the boundary cosmological constant and the derivative of the disk amplitude of the FZZT brane, respectively. It is often convenient to label the FZZT branes by the auxiliary parameter σ, in terms of which

$$x = \cosh \pi b\sigma, \qquad y = \cosh \pi b^{-1}\sigma\,. \tag{1.4}$$

Viewed as a branched cover of the complex x-plane, $\mathcal{M}_{p,q}$ clearly has p different sheets. The surface can be uniformized (i.e. reduced to a single copy of the complex plane) by introducing the parameter

$$z = \cosh \frac{\pi\sigma}{\sqrt{pq}} \tag{1.5}$$

in terms of which $x = T_p(z)$ and $y = T_q(z)$.

Perhaps the most important feature of $\mathcal{M}_{p,q}$ is that it has singularities where

$$F = \partial_x F = \partial_y F = 0\,. \tag{1.6}$$

The singularities are labelled by integers $m = 1, \ldots, p-1$, $n = 1, \ldots, q-1$ satisfying $qm - pn > 0$, and they are located at

$$x_{mn} = (-1)^m \cos\frac{\pi n p}{q}\,, \qquad y_{mn} = (-1)^n \cos\frac{\pi m q}{p}\,. \tag{1.7}$$

In terms of the uniformizing parameter, the singularities are points on $\mathcal{M}_{p,q}$ which correspond to two different values of z:

$$z^{\pm}_{mn} = \cos\frac{\pi(mq \pm np)}{pq}\,. \tag{1.8}$$

The singularities can also be thought of as pinched cycles of a higher-genus surface. Correspondingly, we can define a canonical basis of A and B-cycles on $\mathcal{M}_{p,q}$ labelled by m and n, with A_{mn} (B_{mn}) circling around (passing through) the (m,n) singularity. It turns out that these cycles are closely related to another class of D-branes, the ZZ branes of minimal string theory [16]. The independent ZZ branes are also labelled by integers (m,n) and are in one-to-one correspondence with the singularities of $\mathcal{M}_{p,q}$.

In section 2, we start from the worldsheet description and derive simple formulas for the annulus amplitude between two FZZT branes

$$Z(\sigma|\sigma') = \log\left(\frac{z-z'}{T_p(z)-T_p(z')}\right), \tag{1.9}$$

between an (m,n) ZZ brane and an FZZT brane

$$Z(m,n|\sigma) = \log\left(\frac{z-z^-_{m,n}}{z-z^+_{m,n}}\right), \tag{1.10}$$

and between two ZZ branes labelled by (m,n) and (m',n')

$$Z(m,n|m',n') = \log\frac{(z^+_{m,n}-z^+_{m',n'})(z^-_{m,n}-z^-_{m',n'})}{(z^+_{m,n}-z^-_{m',n'})(z^-_{m,n}-z^+_{m',n'})}. \tag{1.11}$$

Motivated by [40, 41], we take the inverse Laplace transform of $Z(m,n|\sigma)$ with respect to the boundary cosmological constant to obtain the "minisuperspace amplitude" $\Psi_{m,n}(\ell)$ at fixed loop length ℓ. This quantity is a target-space probe of the background (m,n) ZZ brane, and we use the large and small ℓ behavior of $\Psi_{m,n}(\ell)$ to verify that the ZZ branes are located deep in the strong-coupling region $\phi\to+\infty$ of the Liouville direction.

In section 3, we extend the analysis of the deformations of $\mathcal{M}_{p,q}$ given in [31], at the same time providing a geometric interpretation for the ZZ-FZZT annulus amplitude. There are two types of deformations of $\mathcal{M}_{p,q}$: singularity-preserving deformations, which change only the B-cycles of $\mathcal{M}_{p,q}$; and singularity-destroying deformations, which change both the A and B-cycles. In [31], the former class of deformations was considered in detail, and it was shown that they correspond to the addition of local closed-string operators to the worldsheet action. Here we study the space of singularity-destroying deformations, and we argue that this space is spanned by the ZZ brane deformations. The main point is that adding (m,n) ZZ branes to the background deforms the FZZT disk amplitude to leading order by $Z(m,n|\sigma)$, so that the deformation of the curve (1.2) is

$$\delta_{m,n}F \sim U_{p-1}(y)\partial_x Z(m,n|\sigma), \tag{1.12}$$

where $U_{p-1}(\cos\theta) = \frac{\sin(p\theta)}{\sin(\theta)}$ is a Chebyshev polynomial of the second kind. Starting from (1.10) and (1.12), we show that the deformation can be written as a polynomial in x and y

$$\delta_{m,n}F \sim \sum_{r,s} U_{s-1}(x_{mn})U_{r-1}(y_{mn})U_{s-1}(x)U_{p-r-1}(y), \tag{1.13}$$

and we demonstrate that it deforms only the (m, n) A-cycle of the surface, leaving the other A-cycles unchanged. Thus, the effect on $\mathcal{M}_{p,q}$ of inserting background (m, n) ZZ branes is to open up the (m, n) singularity into a regular cycle. In this sense, the ZZ branes "diagonalize" the A-cycle deformations of $\mathcal{M}_{p,q}$.

This raises the question of what closed-string operators diagonalize the B-cycle deformations of $\mathcal{M}_{p,q}$. Focusing on the tachyons $\mathcal{T}_{r,s}$, we find the linear combinations $\tilde{\mathcal{T}}_{m,n}$ that deform only the (m, n) B-cycle of the surface:

$$\tilde{\mathcal{T}}_{m,n} = \sum_{r,s} U_{s-1}(x_{mn}) U_{r-1}(y_{mn}) \mathcal{T}_{r,s}\,. \tag{1.14}$$

We show that these correspond to eigenstates of the ground ring, and that their eigenvalues are precisely the singularities of $\mathcal{M}_{p,q}$.

The similarity between the deformations (1.13) and (1.14) hints at a duality between the ZZ branes and the tachyon eigenstates. In section 4, we attempt to make this duality more precise. We propose that the ZZ branes can be thought of as "wrong-branch" tachyons, i.e. tachyons which violate the usual bound $\alpha < Q/2$ [42]. Our proposal is motivated by the following target-space intuition. In the first quantized description of the theory, the background at large negative ϕ (weak coupling) is determined by solving the Wheeler-deWitt equation [42, 43, 40]. This is a second-order differential equation in ϕ, and as such, it has two linearly independent solutions. The solution which decays in the strong coupling end $\phi \to +\infty$ corresponds to closed string states which satisfy the bound $\alpha < Q/2$. It describes the deformation of the system by closed strings at the weak coupling end $\phi \to -\infty$ where it diverges. It is then natural to ask: what does the other solution correspond to? It satisfies the field equations but does not respect the boundary conditions; it can be taken to decay at the weak coupling end $\phi \to -\infty$ but it diverges at the strong coupling end. As a simple example of a solution which has a singularity, consider the $\frac{1}{r^2}$ solution of the electric field in $3 + 1$ dimensions. The singularity at $r = 0$ means that a charged particle is present there. However, in general, the existence of such a solution cannot be used as an argument for the existence of charged particles. In our case, as in other similar situations in string theory, this background turns out to be sourced by a brane. This is the ZZ-brane. Indeed, the number of distinct ZZ-branes is the same as the number of tachyons in the system [31]. As we will show, appropriate linear combinations of these ZZ-branes source the different tachyons which violate the bound $\alpha < Q/2$ and have the "wrong" ϕ dependence.

Finally, in section 5, we consider in detail the models with $(p, q) = (2, 2k-1)$ and show how many aspects of the general analysis simplify for these models. In appendix A, we study more general annulus amplitudes, and we use these to test the FZZT identification formulas of [31]. Appendix B contains a discussion of the proper normalization of the ZZ boundary states in minimal string theory. We argue that the boundary states should be normalized with an extra minus sign relative to the normalization in Liouville theory alone. This sign leads to the surprising result that the open strings stretched between ZZ and FZZT branes are fermions. (The emergence of fermions in the bosonic string has already been discussed in [44].)

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2. Annulus amplitudes

2.1 FZZT-FZZT annulus amplitude

In this section, we will study the annulus amplitude between two FZZT branes, using the continuum Liouville approach. This was computed in [19]; we quote here the result:

$$Z(\sigma|\sigma') = \frac{1}{p}\sum_{j=1}^{p-1}\int_{-\infty}^{\infty} d\nu \frac{\sin^2(\pi j/p)\cos(\pi\sigma\nu)\cos(\pi\sigma'\nu)}{\nu\sinh(\pi\nu/b)\Big(\cosh(\pi\nu/b)-\cos(\pi j/p)\Big)}\,. \tag{2.1}$$

Here we have implicitly chosen the matter boundary state to be the $(1,1)$ Cardy state (or $a=c=1$ in the notation of [19]). This choice of matter state will be assumed throughout, and to simplify the notation, we will continue to suppress the matter label. We lose no generality in doing so, since FZZT branes with other matter states can be written as a linear combination of branes with matter state $(1,1)$, up to BRST exact terms [31]. (We will discuss the FZZT identifications further in appendix A.) Note that the integral (2.1) has a divergence due to the double pole at $\nu=0$; we will regularize this by replacing the factor $1/\nu$ by $\nu/(\nu^2+\varepsilon^2)$

$$Z(\sigma|\sigma') = \frac{1}{p}\sum_{j=1}^{p-1}\int_{-\infty}^{\infty} d\nu \frac{\nu}{\nu^2+\varepsilon^2}\frac{\sin^2(\pi j/p)\cos(\pi\sigma\nu)\cos(\pi\sigma'\nu)}{\sinh(\pi\nu/b)\Big(\cosh(\pi\nu/b)-\cos(\pi j/p)\Big)}\,. \tag{2.2}$$

In appendix A, the summation over j is performed, leading to

$$Z(\sigma|\sigma') = \int_{-\infty}^{\infty} d\nu \frac{\nu}{\nu^2+\varepsilon^2}\frac{\cos(\pi\sigma\nu)\cos(\pi\sigma'\nu)\sinh(\pi(p-1)\nu/b)}{\sinh(\pi\nu/b)\sinh(\pi p\nu/b)}\,. \tag{2.3}$$

Now let us proceed to evaluate this integral in two steps.

The first step is to close the contour of integration either into the upper or the lower half-plane, so as to pick up the poles at

$$\nu = \pm i\varepsilon \qquad \text{and} \qquad \nu = \frac{ik}{\sqrt{pq}} \quad \text{for } k\in\mathbb{Z} \quad (k\neq 0)\,. \tag{2.4}$$

Our task is slightly complicated by the fact that the particular contour deformation we must use depends on the values of σ and σ'. For instance, if σ and σ' are real and $\sigma>\sigma'>0$, then we can write

$$\cos\pi\sigma\nu = \frac{e^{i\pi\sigma\nu}+e^{-i\pi\sigma\nu}}{2} \tag{2.5}$$

in the integrand of (2.3), in which case the integral over the first (second) exponential can be deformed to pick up the poles in the upper (lower) half-plane. The poles (2.4) with $k=0 \bmod p$ and $k\neq 0 \bmod p$ behave rather differently, leading to two separate sums. Including the pole at $\nu=\pm i\varepsilon$, we obtain

$$\begin{aligned} Z(\sigma|\sigma') =\ & \frac{p-1}{p}\sum_{k=1}^{\infty}\frac{2}{k}e^{-\frac{\pi pk\sigma}{\sqrt{pq}}}\cosh\frac{\pi pk\sigma'}{\sqrt{pq}} - \sum_{\substack{k=1\\ k\neq 0 \bmod p}}^{\infty}\frac{2}{k}e^{-\frac{\pi k\sigma}{\sqrt{pq}}}\cosh\frac{\pi k\sigma'}{\sqrt{pq}} + \\ & + \left(\frac{p-1}{\varepsilon\sqrt{pq}} - (p-1)\frac{\pi\sigma}{\sqrt{pq}} + \mathcal{O}(\varepsilon)\right) \qquad (\sigma>\sigma'>0)\,. \end{aligned} \tag{2.6}$$

The second step in evaluating (2.3) is to drop the divergent piece and evaluate the sums in (2.6). This yields a simple, closed-form expression for the annulus amplitude

$$Z(\sigma|\sigma') = \log\left(\frac{\cosh\frac{\pi\sigma}{\sqrt{pq}} - \cosh\frac{\pi\sigma'}{\sqrt{pq}}}{\cosh\frac{p\pi\sigma}{\sqrt{pq}} - \cosh\frac{p\pi\sigma'}{\sqrt{pq}}}\right). \tag{2.7}$$

The analytic continuation of this expression to all σ, σ' is self-evident. An important check of this formula is that it agrees (up to an overall sign) with the two-macroscopic-loop amplitude obtained in the two-matrix model [45] which generalizes expressions that had been found earlier in special cases [40, 41, 46, 47]. Moreover, one can easily see that (2.7) depends only on the "uniformizing parameter" z defined in (1.5). In terms of z, (2.7) becomes simply

$$Z(\sigma|\sigma') = \log\left(\frac{z-z'}{T_p(z)-T_p(z')}\right) = \log\left(\frac{z-z'}{x-x'}\right). \tag{2.8}$$

Evidently, the annulus amplitude has some rather interesting properties when viewed as a function of z and z'. It is finite when $z = z'$ for generic z, but it diverges when $T_p(z) = T_p(z')$ with $z \neq z'$. It also diverges when $z = z'$ and $U_{p-1}(z) = 0$, which corresponds to the ends of the cuts in the various sheets. In other words, if we think of the Riemann surface as a p-sheeted cover of the complex $x = T_p(z)$ plane parametrized by z, the annulus amplitude is *finite* when $x \to x'$ on the same sheet (except at the ends of the cuts), but *diverges* when $x \to x'$ on different sheets.

The expression (2.8) is closely related to the partition function of two FZZT branes. In the matrix model, the FZZT brane is identified with the macroscopic loop operator $W(x) = \mathrm{Tr}\log(x-M)$. Thus it is represented by an insertion of

$$e^{W(x)} = \det(x-M) \tag{2.9}$$

into the matrix integral. Consider now the two point function $\langle\det(x-M)\det(x'-M)\rangle$. It should be separated into connected and disconnected contributions

$$\langle\det(x-M)\det(x'-M)\rangle = \langle\det(x-M)\rangle\langle\det(x'-M)\rangle e^{Z(x,x')}, \tag{2.10}$$

where the exponent $Z(x,x')$ is given by a sum of connected diagrams

$$Z(x,x') = \sum_{m,n\geq 1}\frac{1}{m!n!}\langle W(x)^m W(x')^n\rangle_c. \tag{2.11}$$

These connected correlation functions are computed in string theory by worldsheets having m boundaries with $\mu_B = x$ and n boundaries with $\mu_B = x'$. To leading order in $1/N$, $Z(x,x')$ is just given by the annulus amplitude (2.8), which leads to

$$\langle\det(x-M)\det(x'-M)\rangle = \langle\det(x-M)\rangle\langle\det(x'-M)\rangle\left(\frac{z-z'}{x-x'} + \mathcal{O}\Big(\frac{1}{N}\Big)\right). \tag{2.12}$$

Clearly the operator $\det(x-M)$ is a boson, and the two point function (2.12) is invariant under the interchange of x and x'. However, comparison with the matrix model suggests that $\det(x-M)$ can be interpreted as creating a *fermionic* probe eigenvalue. This suggests that one should combine $\det(x-M)$ with some kind of cocycle factor to produce a fermionic operator. This might also have the effect of removing the denominator $x-x'$ of (2.12) (see also the discussion at the end of this subsection).

Finally, let us comment briefly on the form of the sum in (2.6). We can view this as a sum over physical states with $k \neq 0 \bmod p$ and unphysical states with $k = 0 \bmod p$. To understand how the latter arise, it is useful to consider the annulus amplitude for fixed boundary length ℓ (i.e. the inverse Laplace transform of the annulus with respect to μ_B and μ'_B). According to [19], this has the form

$$Z(\ell_1, \ell_2) \sim \int_{-\infty}^{\infty} d\nu\, G(\nu) \psi_\nu(\ell_1) \psi_\nu(\ell_2)\,, \tag{2.13}$$

where

$$\psi_\nu(\ell) = K_{2i\nu/b}(\ell) \sqrt{\nu \sinh(2\pi\nu/b)} \tag{2.14}$$

are the normalized wavefunctions and

$$G(\nu) = \sum_{j=1}^{p-1} \frac{\sin^2(\pi j/p)}{\cosh(2\pi\nu/b) - \cos(\pi j/p)} \tag{2.15}$$

is the propagator. Notice how it only has poles at the locations of the physical states $k \neq 0 \bmod p$. This means that when we close the contour to pick up these poles, we will arrive at a sum over only the physical states

$$Z(\ell_1, \ell_2) \sim \sum_{\substack{k=1 \\ k \neq 0 \bmod p}}^{\infty} k \sin(\pi k/p) K_{\frac{k}{p}}(\ell_1) I_{\frac{k}{p}}(\ell_2) \qquad (\ell_1 > \ell_2)\,. \tag{2.16}$$

This form of the ℓ-space annulus was obtained previously for various special cases in [41]. This expression (2.16) is similar to (2.6), but here the unphysical states with $k = 0 \bmod p$ do not contribute. It is only in the Laplace transform to fixed σ and σ' that we obtain the unphysical poles in the propagator. In fact, it is not difficult to see that these poles arise from the $\ell_1,\ \ell_2 \to 0$ part of the integral. Thus we can think of these poles as the contribution to the annulus amplitude from zero-area worldsheets.

It is well-known that zero-area surfaces contribute non-universal terms to the path integral that are analytic in μ [42]. To see this explicitly, we can restore factors of μ (which was set to one) and express the amplitude as a function of μ and μ_B. From (2.6)–(2.8), we see that the unphysical poles sum up to produce the $\log(x-x') = \log(\mu_B - \mu'_B) - \frac{1}{2}\log\mu$ term (up to a contribution proportional to σ) in the annulus amplitude. The $\log\mu$ piece is unimportant for our present discussion. What matters is that the rest is independent of μ. This suggests that we should focus our attention on just the $\log(z-z')$ part of (2.8). (Equivalently, we should differentiate (2.8) with respect to μ and focus on the features of the resulting function.) Then $Z(\sigma|\sigma') = \log(z-z') + \ldots$ is relatively simple — it diverges when the two FZZT branes collide at the same point in $\mathcal{M}_{p,q}$.

We conclude that up to non-universal terms the amplitude is $\log(z-z')$. This discussion also explains the comment above that the denominator $x-x'$ in (2.12) can be removed.

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2.2 ZZ-FZZT and ZZ-ZZ annulus amplitudes

Now it is almost trivial to obtain from (2.8) the ZZ-FZZT annulus amplitude, by using the fact that the ZZ branes can be written as a difference of two FZZT branes (this fact follows from [16, 48, 49] and was made most explicit in [19]). To obtain the correct ZZ-FZZT annulus amplitude, however, we have to take into account one subtlety, concerning the normalization of the ZZ boundary state. This is discussed in appendix B, and the result is that the ZZ branes of minimal string theory come with an extra minus sign relative to the ZZ branes of Liouville theory alone:

$$|m,n\rangle = |\sigma = \sigma_{m,-n}\rangle - |\sigma = \sigma_{m,n}\rangle \tag{2.17}$$

with

$$\sigma_{m,n} = i\left(\frac{m}{b} + nb\right). \tag{2.18}$$

Therefore the annulus amplitude between an (m,n) ZZ brane and an FZZT brane labelled by σ is simply

$$Z(m,n|\sigma) = Z(\sigma_{m,-n}|\sigma) - Z(\sigma_{m,n}|\sigma) = \log\left(\frac{z - z^-_{m,n}}{z - z^+_{m,n}}\right), \tag{2.19}$$

where $z^\pm_{m,n}$ was defined in (1.8). In deriving (2.19), we have used the fact that $T_p(z^+_{m,n}) = T_p(z^-_{m,n})$. The fact that the x dependence cancels out in the ZZ-FZZT annulus amplitude agrees with the idea that this dependence comes from the contribution of zero-area surfaces to the path integral. Since worldsheets with ZZ branes have boundaries of infinite length, they cannot give rise to zero-area surfaces.

As in (2.9)–(2.12) we can exponentiate (2.19) to find the expectation value of the FZZT brane in the presence of a ZZ brane. In the one matrix model

$$\langle \det(x - M)\rangle_{ZZ_n} = \frac{z - z^-_{1,n}}{z - z^+_{1,n}}\left(1 + \mathcal{O}\left(\frac{1}{N}\right)\right). \tag{2.20}$$

This expression vanishes at $z = z^-_{1,n}$ which is the position of the ZZ brane in the first sheet. (It diverges at $z = z^+_{1,n}$ which is in the second sheet.) This is consistent with the interpretation of the ZZ brane as an eigenvalue in the first sheet at $z^-_{1,n}$, which makes the expectation value of $\det(x - M)$ vanish.

Although we have specialized (without loss of generality) to the matter state $(1,1)$ in our discussion so far, it is instructive to consider the ZZ-FZZT annulus amplitudes for other combinations of Liouville and matter labels. One way to do it is by using the expressions in appendix A. Interestingly, the following four choices of ZZ boundary states lead to the same annulus amplitude (2.19):

$$Z(m,n;1,1|\sigma;1,1) = Z(1,1;m,n|\sigma;1,1) = Z(1,n;m,1|\sigma;1,1) = Z(m,1;1,n|\sigma;1,1). \tag{2.21}$$

This degeneracy of ZZ branes was also observed at the level of the disk amplitude in [37]. It can be derived using the boundary state identifications [31, (3.7) and (3.8)].

To complete the discussion, let us consider the ZZ-ZZ annulus amplitude. Using the relation (2.17) once again, it can be obtained from the ZZ-FZZT annulus amplitude (2.19):

$$Z(m,n|m',n') = Z(m,n|\sigma_{m',-n'}) - Z(m,n|\sigma_{m',n'}) = \log\frac{(z^+_{m,n}-z^+_{m',n'})(z^-_{m,n}-z^-_{m',n'})}{(z^+_{m,n}-z^-_{m',n'})(z^-_{m,n}-z^+_{m',n'})}\,. \tag{2.22}$$

Note that the annulus amplitude diverges for $(m,n)=(m',n')$. To get more insight into this divergence we exponentiate (2.22) as above to find the partition function of two ZZ branes

$$\frac{(z^+_{m,n}-z^+_{m',n'})(z^-_{m,n}-z^-_{m',n'})}{(z^+_{m,n}-z^-_{m',n'})(z^-_{m,n}-z^+_{m',n'})}\,. \tag{2.23}$$

Note that unlike (2.20), which has a single zero at $z=z^-_{m,n}$, here we have a double zero for $(m,n)=(m',n')$. This is in agreement with the interpretation of the ZZ branes as eigenvalue. The double zero in (2.23) is the standard double zero for two equal eigenvalues in the matrix model.

We will return to this divergence of (2.22) and the corresponding zero of (2.23) below.

2.3 Target space picture

In section 2.1 we considered the FZZT-FZZT annulus amplitude in ℓ-space [40, 41], which is an important source of physical intuition. Here we consider the analogous transform of the ZZ-FZZT annulus amplitude (2.19). This "minisuperspace amplitude" of the ZZ brane is defined to be the function $\Psi_{m,n}(\ell)$ that satisfies

$$Z(m,n|\sigma) = -\int_0^\infty \frac{d\ell}{\ell}e^{-\ell\cosh\pi b\sigma}\Psi_{m,n}(\ell)\,. \tag{2.24}$$

This identifies the loop length ℓ with the integral of the boundary cosmological constant operator [50],

$$\ell \leftrightarrow \oint e^{b\phi}\,. \tag{2.25}$$

In the minisuperspace approximation, this is just $e^{b\phi_0}$ for some constant ϕ_0, and therefore we can think of $\Psi_{m,n}(\ell = e^{b\phi_0})$ as measuring the fluctuations in the target space fields induced by placing a ZZ brane at $\phi\to+\infty$.

Note that we have avoided calling $\Psi_{m,n}(\ell)$ a wavefunction for the ZZ brane, although its definition is similar to that of the wavefunctions of local closed-string operators. The reason it is inaccurate to think of $\Psi_{m,n}(\ell)$ as a wavefunction is because the ZZ brane is not a local puncture on the worldsheet, but rather a macroscopic boundary. In fact, as we noted above, the boundary of a worldsheet ending on a ZZ brane has infinite length.

Now let us derive a compact formula for $\Psi_{m,n}(\ell)$ and discuss some of its properties. Although in general the inverse Laplace transform can be a difficult operation, here our task is greatly aided by the fact that the annulus amplitude (2.19) satisfies a rather simple differential equation:

$$(x-x_{mn})\partial_x Z(m,n|\sigma) = \frac{4}{p}\sum_{k=1}^{p-1}\frac{\sinh\frac{\pi k\sigma}{\sqrt{pq}}}{\sinh\frac{\pi p\sigma}{\sqrt{pq}}}\,\sin\frac{\pi m(p-k)}{p}\,\sin\frac{\pi n(p-k)}{q}\,. \tag{2.26}$$

Since the Laplace transform of the Bessel function $K_{\frac{k}{p}}(\ell)$ is

$$\int_0^\infty d\ell e^{-\ell \cosh \pi b\sigma} \frac{1}{\pi} K_{\frac{k}{p}}(\ell) \sin \frac{\pi k}{p} = \frac{\sinh \frac{\pi k \sigma}{\sqrt{pq}}}{\sinh \frac{\pi p \sigma}{\sqrt{pq}}} \,, \tag{2.27}$$

we see that (2.26) becomes a first-order differential equation for $\Psi_{m,n}(\ell)$:

$$\frac{\partial}{\partial \ell} \Psi_{m,n}(\ell) = x_{mn} \Psi_{m,n}(\ell) + J_{m,n}(\ell) \,, \tag{2.28}$$

where

$$J_{m,n}(\ell) = \frac{4}{p} \sum_{k=1}^{p-1} K_{\frac{k}{p}}(\ell) \sin \frac{\pi k}{p} \sin \frac{\pi m(p-k)}{p} \sin \frac{\pi n(p-k)}{q} \,. \tag{2.29}$$

This equation can be easily solved, once we impose the boundary condition $\Psi_{m,n}(0) = 0$, which follows from the fact that $\lim_{x \to \infty} \partial_x Z(m, n|\sigma) = 0$. Then we find

$$\Psi_{m,n}(\ell) = \int_0^\ell d\ell' \, \Psi^{(0)}_{m,n}(\ell - \ell') J_{m,n}(\ell') \,, \tag{2.30}$$

where

$$\Psi^{(0)}_{m,n}(\ell) = e^{x_{m,n} \ell} \tag{2.31}$$

is a solution to (2.28) with $J_{m,n} = 0$.

We interpret (2.30) as follows. If the minisuperspace amplitude $\Psi_{m,n}$ were equal to $\Psi^{(0)}_{m,n}$ for all ℓ, the annulus amplitude would be given by

$$Z(m, n|\sigma) \sim -\int_{\Lambda^{-1}}^\infty \frac{d\ell}{\ell} e^{-x\ell} \Psi^{(0)}_{m,n}(\ell) \sim \log\left(\frac{x - x_{m,n}}{\Lambda}\right) , \tag{2.32}$$

where Λ^{-1} is a short ℓ cutoff. (Note that this short ℓ cutoff is not necessary in the Laplace transform of the correct minisuperspace amplitude $\Psi_{m,n}(\ell)$, since this function vanishes at $\ell = 0$.) Given that the FZZT brane corresponds to $\mathrm{Tr} \log(x - X)$ in the dual matrix model, (2.32) amounts to naively replacing the matrix X with the eigenvalue $x_{m,n}$. Then the modification to (2.32) due to the source term $J_{m,n}$ can be interpreted as the effect of interactions between the ZZ brane and other eigenvalues in the Fermi sea.

Since $\Psi_{m,n}(\ell = e^{b\phi_0})$ probes the target-space location of the ZZ brane, we can use the large and small ℓ behavior of $\Psi_{m,n}(\ell)$ to test the idea that the ZZ branes are located at $\phi \to +\infty$ in target space. Let us consider the two limits separately:

1. At small ℓ, we use the fact that $K_{\frac{k}{p}}(\ell) \sim \ell^{-\frac{k}{p}}$ to find

$$\Psi_{m,n}(\ell) \to \frac{2^{1-\frac{1}{p}} (z^-_{m,n} - z^+_{m,n})}{\Gamma\left(\frac{1}{p}\right)} \ell^{\frac{1}{p}} \qquad (\ell \to 0) \,. \tag{2.33}$$

   The vanishing of $\Psi_{m,n}(\ell)$ at $\ell = 0$ implies that the ZZ branes are *not* located at $\phi \to -\infty$ in the weak coupling region of the Liouville direction. Contrast this with the small ℓ behavior of the wavefunctions of local closed-string operators [40] — these always behave as $\ell^{-|\nu|}$ as $\ell \to 0$, since local operators are concentrated at weak-coupling [42].

2. From the explicit form of $J_{m,n}(\ell)$ (2.29), it is not so difficult to show that

$$\int_0^\infty d\ell\, e^{-x_{m,n}\ell} J_{m,n}(\ell) = 1\,, \tag{2.34}$$

which implies that at large ℓ,

$$\Psi_{m,n}(\ell) \to \exp\left(x_{m,n}\ell\right) \qquad (\ell \to \infty)\,. \tag{2.35}$$

Before we interpret (2.35), we must take into account the fact that the FZZT brane dissolves at strong coupling and therefore cannot necessarily probe the ZZ brane located there. To remove the effect of the dissolving FZZT brane, it is reasonable to divide the minisuperspace amplitude by, say, the FZZT disk amplitude $Z(\ell)$. Since this vanishes as $K_\nu(\ell) \sim e^{-\ell}$ as $\ell \to \infty$, the ratio of $\Psi_{m,n}(\ell)$ to $Z(\ell)$ is always exponentially increasing at large ℓ (recall that $|x_{mn}| < 1$). This confirms that the ZZ branes are indeed localized deep in the strong coupling region.

3. Geometric interpretation

3.1 ZZ brane deformations

Having worked out in detail the properties of the ZZ-FZZT annulus amplitude in the previous section, let us now interpret this quantity geometrically. In [31], it was argued that adding (m,n) ZZ branes to the background splits the (m,n) singularity, giving a nonzero value to the integral of $y\,dx$ around the singularity. Now we are in a position to understand this explicitly using the ZZ-FZZT annulus amplitude. The important thing to note is that, to leading order, the change in the FZZT disk amplitude due to the addition of N_{mn} background ZZ branes is measured by the annulus amplitude:

$$\delta_{m,n} Z(\sigma) = g_s N_{m,n} Z(m,n|\sigma)\,. \tag{3.1}$$

Here we must require $g_s N_{mn} \ll 1$ in order for the perturbation expansion to make sense. A deformation of the disk amplitude leads to a deformation of the curve $y = \partial_x Z(\sigma)$,

$$\delta_{m,n} y = \partial_x \delta_{m,n} Z \sim \partial_x Z(m,n|\sigma) = \frac{1}{pU_{p-1}(z)}\left(\frac{1}{z - z^-_{m,n}} - \frac{1}{z - z^+_{m,n}}\right), \tag{3.2}$$

and finally to a deformation of the surface

$$\delta_{m,n} F = pU_{p-1}(y)\delta y \sim \frac{U_{p-1}(T_q(z))}{U_{p-1}(z)}\left(\frac{1}{z - z^-_{m,n}} - \frac{1}{z - z^+_{m,n}}\right). \tag{3.3}$$

It is not too difficult to see that $\delta_{m,n}F$ is a polynomial in z. It turns out that it is also a polynomial in x and y:

$$\delta_{m,n} F \sim -8(-1)^{m+n} \sin\frac{\pi n p}{q} \sin\frac{\pi m q}{p} \sum_{r,s} U_{s-1}(x_{mn}) U_{r-1}(y_{mn}) U_{s-1}(x) U_{p-r-1}(y)\,, \tag{3.4}$$

where the sum runs over the range $1 \le r \le p-1$, $1 \le s \le q-1$, $qr - ps > 0$. The easiest way to see that (3.3) and (3.4) are equal is to notice that both evaluate to the same values at the $(p-1)(q-1)/2$ singularities of $\mathcal{M}_{p,q}$:

$$\delta_{m,n} F(x_{m'n'}, y_{m'n'}) \sim \left(\frac{(-1)^{(q+1)m+(p+1)n} pq}{\sin \frac{\pi np}{q} \sin \frac{\pi mq}{p}} \right) \delta_{m,m'} \delta_{n,n'} \,. \qquad (3.5)$$

This means that, in terms of z, they actually coincide at the $(p-1)(q-1)$ different points $z^{\pm}_{m'n'}$. Since they are both degree $pq-p-q-1$ polynomials in z, the fact that they coincide at $(p-1)(q-1)$ different points implies that they must in fact be equal everywhere.

In the process of showing that (3.3) and (3.4) are equal, we have also shown which singularities of $\mathcal{M}_{p,q}$ are split by $\delta_{m,n}F$. According to (3.5), the (m,n) ZZ brane deformation vanishes at every singularity except the (m,n) singularity. But we also know that a deformation preserves a given singularity if and only if it vanishes on that singularity. Therefore, the effect of adding (m,n) ZZ branes to the background is to split precisely the (m,n) singularity, leaving the others unchanged. In other words, the ZZ branes "diagonalize" the A-cycle deformations of $\mathcal{M}_{p,q}$. This confirms very explicitly the arguments in [31].

Finally, we should discuss the effect of the ZZ branes on the B-cycles of $\mathcal{M}_{p,q}$. Recall that the integral of $y\,dx$ around the (m',n') B-cycle of the surface corresponds to the (m',n') ZZ brane action,

$$\oint_{B_{m',n'}} y\,dx = Z_{m',n'} \,. \qquad (3.6)$$

Adding ZZ branes deforms this period in two ways: first, it deforms the curve y by $\delta_{m,n}y$ given in (3.2); and second, it deforms the contour of integration $B_{m',n'}$. For $(m',n') \neq (m,n)$, the latter effect is subleading, in which case the leading-order deformation to the period is just the annulus amplitude (2.22):

$$\delta_{m,n} \oint_{B_{m',n'}} y\,dx = g_s N_{m,n} Z(m,n|m',n'), \qquad (m',n') \neq (m,n) \,. \qquad (3.7)$$

However, for $(m',n') = (m,n)$ the deformation to the contour of integration $B_{m,n}$ is important, because of the pole in $\delta_{m,n}y$ at x_{mn}. Since the ZZ deformation also splits the singularity at x_{mn} into a branch cut whose width is a positive power of $g_s N_{m,n}$, this has the effect of cutting off the integral $\delta_{m,n} \oint_{B_{m,n}} y\,dx$ at an infinitesimal distance from the pole. Therefore, the diagonal deformation is given by

$$\delta_{m,n} \oint_{B_{m,n}} y\,dx \sim g_s N_{m,n} \log(g_s N_{m,n}) \,. \qquad (3.8)$$

Note that if we had not cut off the integral near the singularity, we would have found a logarithmic divergence. This corresponds to the divergence in the annulus amplitude (2.22) for $(m',n') = (m,n)$. We should also point out that the logarithmic enhancement for $(m',n') = (m,n)$ agrees with the matrix model interpretation of the ZZ branes as fermionic

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eigenvalues, as discussed at the end of section 2.2. If we try to insert multiple ZZ branes at the same point x_{mn}, they will repel one another due to their Fermi statistics, leading to an enhanced backreaction on the surface at x_{mn}.

Since (3.7) and (3.8) are always non-zero, we conclude that the ZZ branes deform all the B-cycles of $\mathcal{M}_{p,q}$. However, there is a sense in which the ZZ branes "diagonalize" the deformations of the B-cycles, since the (m,n) deformation to the (m,n) B-cycle receives a logarithmic enhancement relative to the other cycles.

3.2 Tachyon eigenstate deformations

We have just seen that adding (m,n) ZZ branes deforms the (m,n) A-cycle of $\mathcal{M}_{p,q}$, leaving the other A-cycles unchanged. In this subsection, we will answer a closely related question: what are the deformations that deform just the (m,n) B-cycle of $\mathcal{M}_{p,q}$, leaving the other cycles unchanged? Since these deformations preserve the A-cycles, they must be linear combinations of the singularity-preserving deformations, i.e. they correspond to physical closed-string operators. Then following the reasoning that led to (3.7), we find that the leading-order effect of adding a closed-string operator $\mathcal{V}_{r,s}$ to the worldsheet action is to deform the (m,n) B-cycle by the (m,n) ZZ one-point function of the operator:

$$\delta_{r,s}\oint_{B_{m,n}} y\,dx \sim \langle \mathcal{V}_{r,s}|m,n\rangle \sim \sin\frac{\pi r(mq+np)}{p}\;\sin\frac{\pi s(mq+np)}{q}\,. \tag{3.9}$$

By forming linear combinations of the $\mathcal{V}_{r,s}$, we can clearly arrange for only one particular B-cycle to be deformed.

An especially interesting example is when the $\mathcal{V}_{r,s}$ are the physical tachyon operators $\mathcal{T}_{r,s}$ at ghost number one. To find the linear combinations of tachyons that diagonalize the deformations, recall that the tachyons form a module under the action of the ground ring. Thus we can find linear combinations $\tilde{\mathcal{T}}_{m,n}$ of the tachyons that are eigenstates of the ring,

$$\begin{aligned}\widehat{x}\tilde{\mathcal{T}}_{m,n} &= x_{mn}\tilde{\mathcal{T}}_{m,n}\,,\\ \widehat{y}\tilde{\mathcal{T}}_{m,n} &= y_{mn}\tilde{\mathcal{T}}_{m,n}\,.\end{aligned} \tag{3.10}$$

Here $\widehat{x}=\widehat{\mathcal{O}}_{1,2}/2$ and $\widehat{y}=\widehat{\mathcal{O}}_{2,1}/2$ denote the generators of the ground ring. Since the ZZ branes are also eigenstates of the ring, we find

$$\langle\widehat{x}\tilde{\mathcal{T}}_{m,n}|m',n'\rangle = x_{mn}\langle\tilde{\mathcal{T}}_{m,n}|m',n'\rangle = x_{m'n'}\langle\tilde{\mathcal{T}}_{m,n}|m',n'\rangle \tag{3.11}$$

and similarly for $\widehat{y}$. This is, of course, only possible if and only if

$$\langle\tilde{\mathcal{T}}_{m,n}|m',n'\rangle \propto \delta_{m,m'}\delta_{n,n'}\,. \tag{3.12}$$

In other words, the tachyon eigenstates diagonalize the deformations of the B-cycles of $\mathcal{M}_{p,q}$, with the (m,n) eigenstate deforming only the (m,n) B-cycle.

Note that in this argument, we have implicitly used the fact that the tachyon eigenstates and the ZZ branes have the same set of eigenvalues. This follows from the ring relations $U_{q-1}(\widehat{x}) = U_{p-1}(\widehat{y}) = 0$, together with the additional relation in the tachyon

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module $T_p(\widehat{y}) - T_q(\widehat{x}) = 0$. Since this latter relation is also the equation for $\mathcal{M}_{p,q}$, the eigenvalues in the tachyon module must coincide with the singularities of $\mathcal{M}_{p,q}$, which are of course the eigenvalues of the ZZ branes.

Finally, let us find an explicit formula for the tachyon eigenstates in terms of the usual basis $\mathcal{T}_{r,s}$. This becomes trivial, once one realizes that because the modular transformation matrix S diagonalizes the fusion rules, it also diagonalizes the ring elements and their action on the tachyon module. Since the modular S matrix can be written

$$S_{(m,n),(r,s)} = S_{(m,n),(1,1)} U_{s-1}(x_{mn}) U_{r-1}(y_{mn}) = -\sqrt{\frac{8}{pq}}(-1)^{sm+rn} \sin\frac{\pi snp}{q} \sin\frac{\pi rmq}{p}\,, \tag{3.13}$$

the tachyon eigenstates are given by

$$\tilde{\mathcal{T}}_{m,n} = \sum_{r,s} U_{s-1}(x_{mn}) U_{r-1}(y_{mn}) \mathcal{T}_{r,s}\,. \tag{3.14}$$

Of course, one can also prove this formula in a more straightforward way, by acting on (3.14) with $\widehat{x} - x_{mn}$. For this, one needs to use the ring relations, together with the fact that the tachyons can be written in terms of the ground ring, $\mathcal{T}_{r,s} = U_{s-1}(\widehat{x})U_{r-1}(\widehat{y})\mathcal{T}_{1,1}$. The recurrence relation $xU_{r-1}(x) = U_r(x) + U_{r-2}(x)$ also comes in handy.

To summarize, we have shown that the (m,n) tachyon eigenstate deforms only the (m,n) B-cycle of $\mathcal{M}_{p,q}$, leaving the other B-cycles unchanged. Comparing with our results on the ZZ brane deformations, we see hints of an interesting duality emerging between the tachyon eigenstates and the ZZ branes – they are both eigenstates of the ring, and while the former diagonalizes the B-cycles, the latter diagonalizes the A-cycles of $\mathcal{M}_{p,q}$. We will discuss in detail the evidence for this duality in section 4. In the meantime, let us first wrap up a loose end in the analysis of the ZZ brane deformations.

3.3 General discussion of the deformations

In section 3.1, we found the deformations to the surface due to adding background ZZ branes. But there is one point that we neglected to address. One might have wondered whether the ZZ brane deformations span the entire space of polynomial, singularity-destroying deformations, or whether there are other such deformations that do not correspond to ZZ branes. To answer this question, we need to study the polynomial deformations of $\mathcal{M}_{p,q}$ in slightly more generality.

Consider the space of polynomial, singularity-destroying deformations of $\mathcal{M}_{p,q}$. By this, we mean the quotient ring

$$\mathcal{P}^- \equiv \frac{\mathbb{C}[x,y]}{\{F\} \cup \mathcal{P}^+} \tag{3.15}$$

of polynomials in x and y modulo the original curve $F = T_p(y) - T_q(x)$ and modulo the ideal

$$\mathcal{P}^+ = \{\ U_{p-r-1}(y)U_{s-1}(x) - U_{r-1}(y)U_{q-s-1}(x)\ \} \tag{3.16}$$

of polynomial, singularity-preserving deformations of the curve. We claim that the deformations

$$\tilde{\delta}_{r,s}F = U_{s-1}(x)U_{p-r-1}(y), \qquad 1 \le r \le p-1\,, \qquad 1 \le s \le q-1\,, \qquad qr - ps > 0 \tag{3.17}$$

form a complete basis for $\mathcal{P}^-$. A sketch of the proof is as follows. First, consider the most general polynomial in x and y. Using $F=0$, we can clearly reduce any such polynomial down to one whose degree in y is $\le p-1$. Since the Chebyshev polynomials are a complete set of orthogonal polynomials, this means that

$$U_{k-1}(x)U_{j-1}(y), \qquad 1 \le j \le p, \qquad k \ge 1 \tag{3.18}$$

is a basis for $\mathbb{C}[x,y]/\{F\}$. Now, consider the effect of modding out by $\mathcal{P}^+$. From the form (3.16) of the singularity-preserving deformations, we see that this essentially means that we have the freedom to reflect $(j,k) \to (p-j, q-k)$. If $j=p$ or $k=q$, this is clearly an element of $\mathcal{P}^+$. If $1 \le k \le q-1$, either (3.18) or its reflection is of the form (3.17). Finally, if $k>q$, a reflection turns (3.18) into $U_{q-k-1}(x)U_{p-j-1}(y) = -U_{k-q-1}(x)U_{p-j-1}(y)$. Applying this transformation repeatedly, we can reduce k down to the range $1 \le k \le q-1$, where the previous case holds. This proves that (3.17) form a complete basis for the singularity-destroying deformations of $\mathcal{M}_{p,q}$.

According to (3.17), the number of singularity-destroying deformations is the same as the number of singularities of $\mathcal{M}_{p,q}$. Since there are also this many ZZ brane deformations, these must form an equivalent basis for the space of polynomial, singularity-destroying deformations of $\mathcal{M}_{p,q}$. Indeed, it is immediately obvious from (3.4) that the ZZ brane deformations are related to the basis (3.17) by the action of the modular S-matrix (3.13).

4. ZZ Branes as "wrong-branch" tachyons

We have seen that the ZZ branes and the tachyon eigenstates have very analogous effects on the surface. The former deforms the (m,n) A-cycle of $\mathcal{M}_{p,q}$, while the latter deforms the (m,n) B-cycle. This suggests that the two are in some sense conjugate or dual to one another. In this section, we will try and make this duality more precise.

We propose that the ZZ branes can be thought of as "wrong-branch" tachyons. Before we proceed to discuss the evidence for our proposal, perhaps it will be useful to recall the definition of wrong-branch tachyons. These are tachyons whose Liouville part $e^{2\alpha\phi}$ violates the bound $\alpha < Q/2$ [42]. They are usually excluded on the basis that they do not correspond to local closed-string operators. One way to see this is by studying the semiclassical wavefunctions in the minisuperspace approximation. Tachyons with $\alpha < Q/2$ have wavefunctions $K_\nu(\ell)$ with

$$\nu = \frac{2\alpha - Q}{b} \tag{4.1}$$

that are localized at small ℓ and decay exponentially at $\ell \to \infty$ [40]. This is the expected behavior for a local operator inserted at weak coupling. On the other hand, tachyons with $\alpha > Q/2$ have the opposite behavior. Their wavefunctions are given by $I_{|\nu|}(\ell)$, which vanish at $\ell = 0$ and grow exponentially at large ℓ.

Another way to see that the wrong-branch tachyons do not correspond to local closed-string operators is in the target-space picture, where $\ell \sim e^{b\phi}$. Writing the right-branch wavefunction as $K_\nu(\ell) \sim I_{-|\nu|}(\ell) - I_{|\nu|}(\ell)$, we see that the wavefunction can be thought of as the superposition of an incoming mode $I_{-|\nu|}$ propagating towards $\phi \to +\infty$, and an outgoing mode $I_{|\nu|}$ describing reflection off the Liouville potential $e^{2b\phi}$. Then since the wrong-branch wavefunctions are just $I_{|\nu|}(\ell)$, they describe outgoing modes, with no corresponding incoming mode. Thus they do not describe insertions of local operators in the weak-coupling region. Imagine, however, the effect of placing a ZZ brane at $\phi \to +\infty$. This would presumably emit closed-string radiation in the $\phi \to -\infty$ direction, i.e. it would give rise to outgoing modes with no incoming counterpart. This is our first hint that the ZZ branes are related to the wrong-branch tachyons.

Now let us discuss in detail the evidence for a correspondence between the ZZ branes and the wrong-branch tachyons. First, notice the similarity between the form of the ZZ brane deformations (3.4) and the tachyons (3.14), both of which are eigenstates of the ground ring. This suggests that there is a sense in which the (m, n) ZZ brane is a linear combination of wrong-branch (r, s) tachyons with well-defined KPZ scaling (we will discuss the KPZ scaling below), i.e.

$$|m,n\rangle = \sum_{\substack{r,s \\ qr-ps>0}} U_{s-1}(x_{mn}) U_{r-1}(y_{mn}) |r,s\rangle\rangle \tag{4.2}$$

by analogy with (3.14). (We should think of (4.2) as defining the states $|r, s\rangle\rangle$, i.e. these states are special linear combinations of the usual (m, n) ZZ branes.) We can also provide a simpler explanation for (4.2). Recall that the (m, n) ZZ brane with matter state $(1, 1)$ can also be thought of as a $(1, 1)$ ZZ brane with matter state (m, n). The linear transformation between the (m, n) matter Cardy states and the (r, s) Ishibashi states is essentially the matrix that appears in (4.2).[1] Thus we can think of the states $|r, s\rangle\rangle$ as labelling the different matter Ishibashi states. In this sense, they have well-defined closed-string quantum numbers (e.g. KPZ dimension).

The second piece of evidence for our proposal comes from the ZZ brane minisuperspace amplitudes discussed in section 2.3. There we showed that they vanish at $\ell = 0$ as $\ell^{1/p}$ and grow (relative to the FZZT disk amplitude) exponentially at large ℓ. This agrees with the expected behavior of wrong-branch wavefunctions. We can actually make a much more detailed test of our proposal by expanding the minisuperspace amplitude to higher orders in ℓ at small ℓ. This corresponds to the semiclassical regime (i.e. weak coupling), where we expect to find the Bessel functions $I_{|\nu|}(\ell)$. Indeed, at large x the annulus amplitude has the expansion

$$Z(m,n|\sigma) = -\sum_{k=1}^{\infty} \frac{4}{k} e^{-k\frac{\pi\sigma}{\sqrt{pq}}} \sin\frac{\pi mk}{p} \sin\frac{\pi nk}{q}\,. \tag{4.3}$$

The Laplace transform of this expression can be found by using the formula

$$\int_0^\infty \frac{d\ell}{\ell} e^{-\ell\cosh\tau} I_\nu(\ell) = \frac{1}{\nu} e^{-\nu\tau}\,. \tag{4.4}$$

[1] More precisely, the matter Cardy and Ishibashi states are related via the matrix $\frac{S_{(m,n),(r,s)}}{\sqrt{S_{(1,1),(r,s)}}}$, with S given in (3.13). The difference between this and (4.2) can be absorbed into the normalizations of $|r, s\rangle\rangle$ and $|m, n\rangle$.

From (4.3) and (4.4), we obtain the minisuperspace amplitude $\Psi_{m,n}(\ell)$ at small ℓ as an infinite sum of modified Bessel functions

$$\Psi_{m,n}(\ell) = \frac{4}{p}\sum_{k=1}^{\infty} I_{\frac{k}{p}}(\ell)\sin\frac{\pi m k}{p}\sin\frac{\pi n k}{q}\,. \tag{4.5}$$

The physical interpretation of this expansion is clear: these are the fluctuations in the closed-string modes at $\phi \to -\infty$ due to the insertion of an (m,n) ZZ brane at $\phi \to +\infty$. The fluctuations are all described by the Bessel functions I_ν, and therefore they are all on the wrong-branch.[2]

The third and final piece of evidence for our proposal was alluded to above, namely the KPZ scaling of the "ZZ branes" $|r,s\rangle\rangle$. Again, we can extract this from the small ℓ behavior of the minisuperspace amplitude; if $\Psi_{m,n}(\ell)$ behaves as ℓ^ν as $\ell \to 0$, then the KPZ dimension α of the associated operator is given in (4.1). For instance, the tachyons have $\nu = -(qr-ps)/p$, so $\alpha = \frac{p+q-qr+ps}{2\sqrt{pq}}$, which is the correct formula for the Liouville exponent of the tachyon. According to (4.2) and (3.4), the annulus amplitude of the state $|r,s\rangle\rangle$ with an FZZT brane is

$$\partial_x Z(r,s|\sigma) = \frac{U_{p-r-1}(y)U_{s-1}(x)}{U_{p-1}(y)}\,. \tag{4.6}$$

Expanding this at large σ, we find to leading order that

$$Z(r,s|\sigma) \sim e^{-\left(\frac{qr-ps}{p}\right)\pi b\sigma}\,. \tag{4.7}$$

Therefore, the minisuperspace amplitude behaves as

$$\Psi_{r,s}(\ell) \sim I_{\frac{qr-ps}{p}}(\ell) \sim \ell^{\frac{qr-ps}{p}} \tag{4.8}$$

at small ℓ. Then according to (4.1), the KPZ dimension of the operators $\mathcal{Z}_{r,s}$ is

$$\alpha_{r,s} = \frac{p+q+qr-ps}{2\sqrt{pq}} \tag{4.9}$$

which is precisely the expected dimension for a wrong-branch (r,s) tachyon.

5. A closer look at the $(p,q) = (2,2k-1)$ theories

Here we will discuss in detail the models with $(p,q) = (2,2k-1)$. Apart from serving as concrete examples for the general analysis above, these models also contain many additional simplifications. For instance, it is easy to see that the minisuperspace amplitude studied above becomes particularly simple when $p = 2$. The source term $J_{1,n}$ in (2.29) is not a sum of many Bessel functions, but is given by a single function $K_{\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2z}}e^{-z}$. Also, we can set $m = 1$ for the (m,n) ZZ-brane, since the independent set of ZZ-branes is spanned by $(m,n) = (1,n)$ with $n = 1,\cdots,(q-1)/2$ [31]. Then the minisuperspace amplitude of the

[2] It is also nice that the terms with $k = 0 \mod p,q$ do not contribute to the expansion. This is consistent with the BRST analysis of Lian and Zuckerman [51].

$(1, n)$ ZZ brane (2.30) is just

$$\Psi_{1,n}(\ell) = \exp(x_{1,n}\,\ell)\,\mathrm{Erf}\left(\sin\frac{\pi n}{q}\,(2\ell)^{\frac{1}{2}}\right), \tag{5.1}$$

where $\mathrm{Erf}(z)$ is the error function defined by

$$\mathrm{Erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^z dx\, e^{-x^2} \tag{5.2}$$

and $x_{1,n}$ is given by

$$x_{1,n} = -\cos\frac{2\pi n}{q}\,. \tag{5.3}$$

One of the reasons for the simplification at $p = 2$ is that the $(2, q)$ theory is described by a one-matrix model. Accordingly, the Riemann surface $\mathcal{M}_{2,q}$ for this theory is a double cover of the x-plane [31]

$$F(x, y) = T_q(x) + 1 - 2y^2 = 0\,. \tag{5.4}$$

On the x-plane, there is a square root cut along $x \leq -1$ and the points $x = x_{1,n}$ are the singularities of the curve (5.4).

The minisuperspace amplitude (5.1) can be interpreted more geometrically if we write it as a line integral on this curve. For this purpose, it is useful to rewrite the amplitude (2.19) in terms of the x-coordinate

$$Z(1, n|\sigma) = \log\left(\frac{\sqrt{x+1} - \sqrt{x_{1,n}+1}}{\sqrt{x+1} + \sqrt{x_{1,n}+1}}\right), \tag{5.5}$$

where we substituted the following relations valid for the $p = 2$ case

$$z = \sqrt{\frac{x+1}{2}}\,, \qquad z^{\pm}_{1,n} = \mp\sin\frac{\pi n}{q} = \mp\sqrt{\frac{x_{1,n}+1}{2}}\,. \tag{5.6}$$

Then the minisuperspace amplitude in question is just an inverse Laplace transform of

$$\partial_x Z(1, n|\sigma) = \frac{\sqrt{x_{1,n}+1}}{\sqrt{x+1}(x - x_{1,n})}\,. \tag{5.7}$$

The usual prescription for the inverse Laplace transform of a function $f(x)$ is to integrate $e^{\ell x} f(x)$ along a contour parallel to the imaginary x-axis and to the right of all the singularities of $f(x)$. Applying this to (5.7) results in the following formula for $\Psi_{1,n}(\ell)$:

$$\Psi_{1,n}(\ell) = \oint_C \frac{dx}{2\pi i}\, e^{\ell x}\, \frac{\sqrt{x_{1,n}+1}}{\sqrt{x+1}(x - x_{1,n})}\,, \tag{5.8}$$

where C denotes a contour surrounding the cut $x \leq -1$ and the pole $x = x_{1,n}$.

As discussed in [39], we can gain more insight into the effect of ZZ-brane by introducing the density $\rho_{1,n}(x)$ as

$$\Psi_{1,n}(\ell) = \int_{-\infty}^{\infty} dx\, e^{\ell x} \rho_{1,n}(x)\,. \tag{5.9}$$

$\rho_{1,n}(x)$ can be thought of as the deformation of the eigenvalue density due to the extra eigenvalue introduced at $x = x_{1,n}$. According to (5.8), it is given by the imaginary part of $-\partial_x Z(1,n|\sigma)$ across the real x-axis. The explicit form of $\rho_{1,n}(x)$ is found to be

$$\rho_{1,n}(x) = \delta(x - x_{1,n}) - \theta(-1-x)\frac{1}{\pi}\frac{\sqrt{1+x_{1,n}}}{\sqrt{-1-x}(x_{1,n}-x)}\,. \tag{5.10}$$

Here $\theta(-1-x)$ denotes the step function, which is 1 for $x \leq -1$ and 0 for $x > -1$. The δ-function in (5.10) represents the extra eigenvalue sitting at $x = x_{1,n}$, while the second term in (5.10) can be interpreted as the backreaction to the Fermi sea in putting an eigenvalue at $x = x_{1,n}$ [39].[3]

Note that the coefficient of the δ-function in (5.10) is exactly one. It was emphasized in [39] that this is a consequence of the Cardy condition. Another interesting feature of (5.10) is that

$$\int_{-\infty}^{\infty} dx\, \rho_{1,n}(x) = 0\,. \tag{5.11}$$

This follows from the definition (5.9) of $\rho_{1,n}(x)$ and the fact that the minisuperspace amplitude $\Psi_{1,n}(\ell)$ vanishes at $\ell = 0$. Of course, it can also be checked by the direct integration of (5.10). The relation (5.11) suggests the picture that exactly one eigenvalue is removed from the Fermi sea $x \leq -1$ and placed at the point $x = x_{1,n}$ which is an extremum of the effective potential.

For the $p = 2$ models, the ZZ brane deformations are also extremely simple. Substituting $p = 2$ and (5.6) into (3.3), we find

$$\delta_{1,n}F \sim \frac{y}{(x - x_{1,n})\sqrt{x+1}}\,. \tag{5.12}$$

Remembering that the curve $y(x)$ takes the form

$$y^2 = 2^{q-2}(x+1)\prod_{l=1}^{\frac{q-1}{2}}(x - x_{1,l})^2\,, \tag{5.13}$$

we find

$$\delta_{1,n}F \sim \prod_{l \neq n}(x - x_{1,l})\,. \tag{5.14}$$

Note that the degree of the polynomial (5.14) is $\frac{q-3}{2}$. In particular, $\delta F(x,y)$ is a constant for the pure gravity case $(p,q) = (2,3)$ [39]. From (5.14), it is obvious that the $(1,n)$ ZZ brane deformation vanishes at every singularity except $x = x_{1,n}$; therefore it splits only the $(1,n)$ singularity. Using (5.12), or equivalently

$$\delta_{1,n}y \sim \frac{1}{(x - x_{1,n})\sqrt{x+1}}\,, \tag{5.15}$$

[3] Recall also our interpretation below (2.32) of the source term $J_{1,n}$ as encoding the effect of interactions between the eigenvalue and the Fermi sea.

we can also make very concrete the discussion of the ZZ brane B-cycle deformations at the end of section 3.1. In particular, it is immediately clear from (5.15) that the integral $\delta_{1,n}\oint_{B_{1,n'}} y$ receives a logarithmic enhancement for $n' = n$, since in this case the integral runs between $x \sim -1$ and the branch point near the pole at $x = x_{1,n}$.

The ZZ brane deformations (5.14) have a nice interpretation in the one-matrix model. Recall that at infinite N, before the double-scaling limit, the matrix model curve can be written as

$$y^2 = V'(x)^2 + f(x)\,, \tag{5.16}$$

where $V(x)$ is the matrix model potential of minimal degree $\frac{q+3}{2}$ and $f(x)$ is a polynomial of degree $\frac{q-1}{2}$. The main effect of the double-scaling limit is to scale away the leading term in (5.16), reducing the degree of y^2 from $q+1$ to q. Now let us consider the relevant deformations of the curve (5.16). We can either deform the potential or $f(x)$. Allowing for shifts and rescalings of x, $V'(x)$ has $\frac{q-1}{2}$ free parameters. These correspond to the $\frac{q-1}{2}$ tachyon deformations. Since y^2 has $q-1$ free parameters modulo shifts and rescalings of x, this leaves $\frac{q-1}{2}$ deformations to $f(x)$; these are of degree at most $\frac{q-3}{2}$. Comparing with the degree of the ZZ brane deformations (5.14), we conclude that adding ZZ branes amounts to deforming $f(x)$ while keeping the potential fixed. Since $f(x)$ determines how the eigenvalues are distributed among the extrema of $V(x)$, this confirms that adding ZZ branes corresponds to shifting eigenvalues from the Fermi sea to the other extrema of the potential.

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A. Other annulus amplitudes

In this appendix, we will study the FZZT-FZZT annulus amplitudes with more general matter states. These will provide a highly non-trivial test of the FZZT identification formulas of [31],

$$|\sigma; k, l\rangle = \sum_{m'=-(k-1),2}^{k-1} \sum_{n'=-(l-1),2}^{l-1} |\sigma + \frac{i(m'q+n'p)}{\sqrt{pq}}; 1, 1\rangle \tag{A.1}$$

that relate FZZT branes with general matter state to FZZT branes with matter state $(1,1)$.

The annulus amplitude with general matter boundary conditions was derived in [19] using the continuum Liouville approach:

$$Z(\sigma,a|\sigma',a') = \frac{1}{p}\sum_{j=1}^{p-1}\int_{-\infty}^{\infty} d\nu \frac{\sin(\pi aj/p)\sin(\pi a'j/p)\cos(\pi\sigma\nu)\cos(\pi\sigma'\nu)}{\nu\sinh(\pi\nu/b)\Big(\cosh(\pi\nu/b)-\cos(\pi j/p)\Big)} \tag{A.2}$$

where $a, a'(= 1,\cdots,p-1)$ label the nodes (or "heights") of the A_{p-1} Dynkin diagram [52]. Note that the matter boundary condition with height a corresponds to the minimal model Cardy state $|k,l\rangle = |a,1\rangle$. We wish to simplify the annulus amplitudes (A.2) as in section 2.1. First, we evaluate the sum over j in (A.2) using the identity[4]

$$\frac{1}{p}\sum_{j=1}^{p-1}\frac{\sin(\pi aj/p)\sin(\pi a'j/p)}{\cosh(\pi\nu/b)-\cos(\pi j/p)} = \frac{\sinh(\pi a\nu/b)\sinh(\pi(p-a')\nu/b)}{\sinh(\pi\nu/b)\sinh(\pi p\nu/b)}, \qquad (a,a'\in\mathbb{Z}^+,\ a\le a')\,. \tag{A.3}$$

Then the annulus amplitude (A.2) becomes

$$Z(\sigma,a|\sigma',a') = \int_{-\infty}^{\infty} d\nu \frac{\cos(\pi\sigma\nu)\cos(\pi\sigma'\nu)\sinh(\pi a\nu/b)\sinh(\pi(p-a')\nu/b)}{\nu\sinh^2(\pi\nu/b)\sinh(\pi p\nu/b)}, \qquad (a\le a')\,. \tag{A.4}$$

The integral can be evaluated by regularizing

$$\frac{1}{\nu}\to\frac{\nu}{\nu^2+\varepsilon^2} \tag{A.5}$$

and closing the contour of ν-integral as in section 2.1. For $a, a' = 1, p-1$ this yields (up to an additive constant)

$$\begin{aligned} Z(\sigma,1|\sigma',1) &= Z(\sigma,p-1|\sigma',p-1) = \log\left(\frac{z-z'}{x-x'}\right), \\ Z(\sigma,1|\sigma',p-1) &= -\log(z+z')\,. \end{aligned} \tag{A.6}$$

In general, we find a somewhat more complicated formula

$$\begin{aligned} Z(\sigma,a|\sigma',a') &= -\sum_{k=1}^{\infty}\frac{2}{k}U_{a-1}(\cos\frac{\pi k}{p})U_{a'-1}(\cos\frac{\pi k}{p})e^{-\frac{\pi k\sigma}{\sqrt{pq}}}\cosh\frac{\pi k\sigma'}{\sqrt{pq}} + \\ &\quad + a\sum_{k=1}^{\infty}\frac{2}{k}(-1)^{(a+a')k}e^{-\frac{\pi pk\sigma}{\sqrt{pq}}}\cosh\frac{\pi pk\sigma'}{\sqrt{pq}} + \left(\frac{a(p-a')}{\varepsilon\sqrt{pq}} - a(p-a')\frac{\pi\sigma}{\sqrt{pq}}\right) \\ &\quad (\sigma>\sigma'>0,\ a\le a')\,. \end{aligned} \tag{A.7}$$

Now let us compare (A.7) with the result of using the FZZT identifications (A.1). We define

$$\tilde Z(\sigma,a|\sigma',a') \equiv \sum_{m=-(a-1),2}^{a-1}\ \sum_{m'=-(a'-1),2}^{a'-1} Z\left(\sigma+\frac{im}{b},1|\sigma'+\frac{im'}{b},1\right). \tag{A.8}$$

[4]To prove this identity, notice that both sides are meromorphic functions of ν with the same poles and residues. Therefore they differ by a holomorphic function $f(\nu)$. Since both the LHS and the RHS tend to zero as $\nu\to+\infty$ away from the imaginary ν axis, $f(\nu)$ must be identically zero.

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Then one can show that

$$
\begin{aligned}
Z(\sigma,a|\sigma',a') - \tilde{Z}(\sigma,a|\sigma',a') &= -\frac{a(a'-1)p}{\varepsilon\sqrt{pq}} + a(a'-1)p\frac{\pi\sigma}{\sqrt{pq}} - \\
&\quad - a(a'-1)\sum_{k=1}^{\infty}\frac{2}{k}(-1)^{(a+a')k}e^{-\frac{\pi pk\sigma}{\sqrt{pq}}}\cosh\frac{\pi pk\sigma'}{\sqrt{pq}} \\
&= -\frac{a(a'-1)p}{\varepsilon\sqrt{pq}} + a(a'-1)\log\left(2(x-(-1)^{a+a'}x')\right). \\
&\quad (a \le a')
\end{aligned}
\tag{A.9}
$$

In other words, the actual annulus amplitude (A.7) differs from the result of applying the FZZT identifications by a sum over the unphysical states with $k = 0 \mod p$. According to the discussion at the end of section 2.1, we expect this discrepancy to be non-universal and analytic in μ. Indeed, if we restore the powers of μ by writing $x = \mu_B/\sqrt{\mu}$ and similarly for x', and in addition make the correspondence[5]

$$
\frac{1}{\varepsilon} \to \frac{1}{2b}\log\left(\frac{\Lambda}{\mu}\right),
\tag{A.10}
$$

then (A.9) becomes

$$
Z(\sigma,a|\sigma',a') - \tilde{Z}(\sigma,a|\sigma',a') = a(a'-1)\log\Big(2(\mu_B - (-1)^{a+a'}\mu'_B)\Big) - \frac{1}{2}a(a'-1)\log\Lambda
\tag{A.11}
$$

i.e. the μ dependence drops out all together. Therefore, we have found that the FZZT identifications predict the correct annulus amplitudes up to terms analytic in μ. Since these analytic terms are regularization dependent, the FZZT identifications in some sense provide an alternate regularization of the annulus amplitude.

B. Normalization of the ZZ boundary states

In this appendix, we will discuss the proper normalization of the ZZ boundary states in minimal string theory. We will be particularly interested in the relative normalization of the ZZ and FZZT branes. In Liouville CFT, this normalization is fixed to be [16]

$$
|m,n\rangle_{\text{Liouville}} = |\sigma_{m,n}\rangle_{\text{Liouville}} - |\sigma_{m,-n}\rangle_{\text{Liouville}}
\tag{B.1}
$$

by the requirement that the ZZ-FZZT annulus amplitude must be given by a trace over open-string states stretched between the two branes, with every state contributing $+1$ to the partition function. As we will see below, in minimal string theory the relative normalization can be fixed by target space considerations, and it is different from (B.1).

[5] This correspondence is motivated by the fact that the divergence as $\varepsilon \to 0$ is precisely an infinite volume divergence, while $\frac{1}{2b}\log\left(\frac{\Lambda}{\mu}\right)$ is the usual expression for the "volume" of the Liouville direction cut off by the Liouville wall at $\phi \to +\infty$.

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Consider first the natural normalization of the FZZT boundary state in Liouville theory. The one point function of the cosmological constant on the disk in this state, with a $(1,1)$ boundary state for matter, is given by[6] [14, 16, 22]:

$$\left\langle e^{2b\phi}(0)\right\rangle = -A_M \frac{2^{\frac{1}{4}}}{\sqrt{\pi}}\sqrt{\frac{q}{p}}\,\frac{\Gamma(\frac{p}{q})}{\Gamma(\frac{q}{p})}\frac{1}{\sin\frac{q\pi}{p}}(\sqrt{\mu})^{q/p-1}\cosh\left(b-\frac{1}{b}\right)\pi\sigma\,. \tag{B.2}$$

A_M is the contribution to the one-point function from the matter sector (we assume the $(1,1)$ matter state):

$$A_M = \sqrt{|S_{(1,1),(1,1)}|} = \left(\frac{8}{pq}\right)^{\frac{1}{4}}\left|\sin\frac{\pi(q-p)}{p}\sin\frac{\pi(q-p)}{q}\right|^{\frac{1}{2}}. \tag{B.3}$$

Here $S_{(1,1),(1,1)}$ is an element of the modular S matrix (3.13). It is always positive for the unitary models ($q=p+1$), so in these cases the absolute value in (B.3) can be omitted. For some non-unitary theories, $S_{(1,1),(1,1)}$ is negative; in these cases the absolute value in (B.3) ensures that the one point function (B.2) is real. We will see momentarily that this choice of the phase of the boundary state is natural from the target space point of view as well.

Integrating (B.2) with respect to $-\mu$ holding $\mu_B = \sqrt{\mu}\cosh\pi b\sigma$ fixed, and remembering the rescaling of μ by $\pi\gamma(b^2)$, we find the FZZT disk amplitude:

$$Z(\sigma) = -A_M\frac{2^{\frac{5}{4}}}{\pi^{\frac{3}{2}}}\frac{\sqrt{pq}}{p+q}\frac{\Gamma(1-\frac{p}{q})}{\Gamma(\frac{q}{p})}\frac{1}{\sin\frac{q\pi}{p}}(\sqrt{\mu})^{q/p+1}\left(\sinh\pi b\sigma\sinh\frac{\pi\sigma}{b} - b^2\cosh\pi b\sigma\cosh\frac{\pi\sigma}{b}\right) \tag{B.4}$$

We can now check that the choice of phase of the boundary state implicit in (B.4) is the one natural from the target space point of view. As discussed in [31], the FZZT brane can be identified with the effective potential of a probe eigenvalue in the double scaling limit $Z(\sigma) = -\frac{1}{2}V_{\rm eff}(x)$. Thus, for example, we expect it to be unbounded from below for unitary theories. It is easy to check that for $q=p+1$, (B.4) indeed has the property that as $\sigma\to\infty$ (or equivalently $x\to\infty$ (1.4)),

$$Z(\sigma) \sim C\exp\left(\pi\sigma\left(b+\frac{1}{b}\right)\right) \tag{B.5}$$

with C a positive constant. Hence the effective potential $V_{\rm eff}\to-\infty$ in this limit, in agreement with the original construction of the double scaling limit [5]–[8].

As another check on the phase of (B.4), consider the case $(p,q)=(2,2k-1)$. In this case, one again finds the behavior (B.5), with $C_k\sim(-1)^k$. Thus, the effective potential is not bounded from below for even k and goes to $+\infty$ as $x\to\infty$ for odd k. This, again, is consistent with the known results from double scaled matrix models. Models with k even (such as pure gravity, corresponding to the case $k=2$) correspond to matrix integrals with potentials that are not bounded from below, whereas for k odd the potential is bounded from below and the matrix integral is well defined.

After understanding the natural phase of the FZZT boundary state, we move on to the ZZ branes. These branes give non-perturbative instanton corrections to string amplitudes. One can check that this implies the ZZ boundary states should be defined with an extra

[6]We use the normalization given in [16]; it differs by a minus sign from [14].

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minus sign relative to the CFT definition (B.1):

$$|m,n\rangle = |\sigma_{m,-n}\rangle - |\sigma_{m,n}\rangle \tag{B.6}$$

As an example, consider again the unitary case $q = p+1$. Using the modified prescription (B.6), one finds

$$\begin{aligned} Z_{m,n} &= Z(\sigma_{m,-n}) - Z(\sigma_{m,n}) \\ &= -A_M \frac{2^{\frac{9}{4}}}{\pi^{\frac{3}{2}}} \frac{\sqrt{pq}}{p+q} \frac{\Gamma(1-\frac{p}{q})}{\Gamma(\frac{q}{p})} \frac{1}{\sin\frac{\pi}{p}} (\sqrt{\mu})^{q/p+1} \sin\frac{m\pi}{p} \sin\frac{n\pi}{p+1} < 0\,. \end{aligned} \tag{B.7}$$

As indicated in (B.7), all $Z_{m,n}$ are negative in this case, in agreement with the fact that the leading non-perturbative corrections, which go like $\exp(Z_{m,n})$, give rise to $\exp(-1/g_s)$ instanton effects.

For non-unitary models, (B.6) implies that some of the ZZ branes have in general positive Z_{nm} or negative V_{eff}. By the usual rules of D instantons this implies that in these cases there are non-perturbative effects that go like $e^{+\mathcal{O}(1/g_s)}$ instead of $e^{-\mathcal{O}(1/g_s)}$. The existence of such catastrophic non-perturbative effects seems to be related to the fact that in non-unitary theories the perturbative vacuum is an unstable critical point of the effective potential V_{eff}, and there are nearby lower critical points. For example, when $p = 2$, one can show [31] that V_{eff} has a number of local minima (corresponding to the $(1,n)$ ZZ branes with n odd) with energies *below* the Fermi level. Therefore the perturbative vacuum is unstable to eigenvalues tunneling to these local minima. It is important to emphasize that this phenomenon is separate from the question of whether the matrix model potential is bounded from below at infinity.

One interesting consequence of (B.6) is that there is an extra minus sign in the ZZ-FZZT annulus, relative to Liouville theory alone. This means that the open strings stretched between ZZ and FZZT branes are *fermions* in minimal string theory! It is, of course, very surprising to find fermions emerging out of bosonic string theory. But this interpretation is well-supported by the matrix model, where the FZZT branes are described by the determinant operator $\det(x-M)$. We can also write this determinant as a Grassman integral over N complex fermions ψ_i

$$\det(x-M) = \int d\psi^\dagger d\psi \; e^{\psi^\dagger (x-M)\psi}\,. \tag{B.8}$$

In the one-matrix model, we interpret the matrix M as describing the (bosonic) open strings stretched between the N ZZ branes in the Fermi sea. Therefore, the ψ_i represent the fermionic open strings stretched between the FZZT brane and the N ZZ branes. They are complex because of the two different orientations of these open strings. The parameter x which labels the FZZT brane and corresponds to the boundary cosmological constant appears here as a mass term for the fermions. It can be thought of as representing the length of the corresponding open strings.

References

[1] F. David, *Planar diagrams, two-dimensional lattice gravity and surface models*, *Nucl. Phys.* **B 257** (1985) 45.

[2] V.A. Kazakov, *Bilocal regularization of models of random surfaces*, *Phys. Lett.* **B 150** (1985) 282.

[3] V.A. Kazakov, A.A. Migdal and I.K. Kostov, *Critical properties of randomly triangulated planar random surfaces*, *Phys. Lett.* **B 157** (1985) 295.

[4] J. Ambjørn, B. Durhuus and J. Fröhlich, *Diseases of triangulated random surface models and possible cures*, *Nucl. Phys.* **B 257** (1985) 433.

[5] M.R. Douglas and S.H. Shenker, *Strings in less than one-dimension*, *Nucl. Phys.* **B 335** (1990) 635.

[6] D.J. Gross and A.A. Migdal, *Nonperturbative two-dimensional quantum gravity*, *Phys. Rev. Lett.* **64** (1990) 127.

[7] E. Brezin and V.A. Kazakov, *Exactly solvable field theories of closed strings*, *Phys. Lett.* **B 236** (1990) 144.

[8] M.R. Douglas, *Strings in less than one-dimension and the generalized K-D-V hierarchies*, *Phys. Lett.* **B 238** (1990) 176.

[9] P.H. Ginsparg and G.W. Moore, *Lectures on 2D gravity and 2D string theory*, `hep-th/9304011`.

[10] P. Di Francesco, P.H. Ginsparg and J. Zinn-Justin, *2D gravity and random matrices*, *Phys. Rept.* **254** (1995) 1 [`hep-th/9306153`].

[11] H. Dorn and H.J. Otto, *Some conclusions for noncritical string theory drawn from two and three point functions in the Liouville sector*, `hep-th/9501019`.

[12] A.B. Zamolodchikov and A.B. Zamolodchikov, *Structure constants and conformal bootstrap in Liouville field theory*, *Nucl. Phys.* **B 477** (1996) 577 [`hep-th/9506136`].

[13] J. Teschner, *On the Liouville three point function*, *Phys. Lett.* **B 363** (1995) 65 [`hep-th/9507109`].

[14] V. Fateev, A.B. Zamolodchikov and A.B. Zamolodchikov, *Boundary Liouville field theory, I. Boundary state and boundary two-point function*, `hep-th/0001012`.

[15] J. Teschner, *Remarks on Liouville theory with boundary*, `hep-th/0009138`.

[16] A.B. Zamolodchikov and A.B. Zamolodchikov, *Liouville field theory on a pseudosphere*, `hep-th/0101152`.

[17] B. Ponsot and J. Teschner, *Boundary Liouville field theory: boundary three point function*, *Nucl. Phys.* **B 622** (2002) 309 [`hep-th/0110244`].

[18] J. McGreevy and H. Verlinde, *Strings from tachyons: the $c = 1$ matrix reloaded*, *J. High Energy Phys.* **12** (2003) 054 [`hep-th/0304224`].

[19] E.J. Martinec, *The annular report on non-critical string theory*, `hep-th/0305148`.

[20] I.R. Klebanov, J. Maldacena and N. Seiberg, *D-brane decay in two-dimensional string theory*, *J. High Energy Phys.* **07** (2003) 045 [`hep-th/0305159`].

JHEP08(2004)026

[21] J. McGreevy, J. Teschner and H. Verlinde, *Classical and quantum D-branes in 2D string theory*, *J. High Energy Phys.* **01** (2004) 039 [hep-th/0305194].

[22] S.Y. Alexandrov, V.A. Kazakov and D. Kutasov, *Non-perturbative effects in matrix models and D-branes*, *J. High Energy Phys.* **09** (2003) 057 [hep-th/0306177].

[23] I.K. Kostov, B. Ponsot and D. Serban, *Boundary liouville theory and 2D quantum gravity*, *Nucl. Phys.* **B 683** (2004) 309 [hep-th/0307189].

[24] M.R. Douglas et al., *A new hat for the $c = 1$ matrix model*, hep-th/0307195.

[25] T. Takayanagi and N. Toumbas, *A matrix model dual of type 0B string theory in two dimensions*, *J. High Energy Phys.* **07** (2003) 064 [hep-th/0307083].

[26] I.R. Klebanov, J. Maldacena and N. Seiberg, *Unitary and complex matrix models as 1-D type 0 strings*, hep-th/0309168.

[27] J. Teschner, *On boundary perturbations in Liouville theory and brane dynamics in noncritical string theories*, *J. High Energy Phys.* **04** (2004) 023 [hep-th/0308140].

[28] J.L. Karczmarek and A. Strominger, *Matrix cosmology*, *J. High Energy Phys.* **04** (2004) 055 [hep-th/0309138].

[29] S. Alexandrov, *(m,n) ZZ branes and the $c = 1$ matrix model*, hep-th/0310135.

[30] C.V. Johnson, *Non-perturbative string equations for type 0A*, *J. High Energy Phys.* **03** (2004) 041 [hep-th/0311129].

[31] N. Seiberg and D. Shih, *Branes, rings and matrix models in minimal (super)string theory*, *J. High Energy Phys.* **02** (2004) 021 [hep-th/0312170].

[32] D. Gaiotto and L. Rastelli, *A paradigm of open/closed duality: Liouville D-branes and the Kontsevich model*, hep-th/0312196.

[33] D.J. Gross and J. Walcher, *Non-perturbative RR potentials in the c(hat) = 1 matrix model*, *J. High Energy Phys.* **06** (2004) 043 [hep-th/0312021].

[34] S. Gukov, T. Takayanagi and N. Toumbas, *Flux backgrounds in 2D string theory*, *J. High Energy Phys.* **03** (2004) 017 [hep-th/0312208].

[35] I.K. Kostov, *Boundary ground ring in 2D string theory*, *Nucl. Phys.* **B 689** (2004) 3 [hep-th/0312301].

[36] S. Alexandrov, *Thermodynamics and quantum theory of tensionless noncommutative p-branes*, *J. High Energy Phys.* **05** (2004) 025 [hep-th/0403116].

[37] V.A. Kazakov and I.K. Kostov, *Instantons in non-critical strings from the two-matrix model*, hep-th/0403152.

[38] S.R. Das, J.L. Davis, F. Larsen and P. Mukhopadhyay, *Particle production in matrix cosmology*, *Phys. Rev.* **D 70** (2004) 044017 [hep-th/0403275].

[39] M. Hanada et al., *Loops versus matrices: the nonperturbative aspects of noncritical string*, *Prog. Theor. Phys.* **112** (2004) 131 [hep-th/0405076].

[40] G.W. Moore, N. Seiberg and M. Staudacher, *From loops to states in 2D quantum gravity*, *Nucl. Phys.* **B 362** (1991) 665.

[41] G.W. Moore and N. Seiberg, *From loops to fields in 2D quantum gravity*, *Int. J. Mod. Phys.* **A 7** (1992) 2601.

JHEP08(2004)026

[42] N. Seiberg, *Notes on quantum Liouville theory and quantum gravity*, *Prog. Theor. Phys. Suppl.* **102** (1990) 319.

[43] J. Polchinski, *Remarks on the Liouville field theory*, UTTG-19-90, presented at Strings '90 Conf., College Station, TX, Mar 12-17, 1990.

[44] J.R. David, S. Minwalla and C. Núñez, *Fermions in bosonic string theories*, *J. High Energy Phys.* **09** (2001) 001 [hep-th/0107165].

[45] J.M. Daul, V.A. Kazakov and I.K. Kostov, *Rational theories of 2D gravity from the two matrix model*, *Nucl. Phys.* **B 409** (1993) 311 [hep-th/9303093].

[46] J. Ambjørn, J. Jurkiewicz and Y.M. Makeenko, *Multiloop correlators for two-dimensional quantum gravity*, *Phys. Lett.* **B 251** (1990) 517.

[47] I.K. Kostov, *Strings with discrete target space*, *Nucl. Phys.* **B 376** (1992) 539 [hep-th/9112059].

[48] K. Hosomichi, *Bulk-boundary propagator in Liouville theory on a disc*, *J. High Energy Phys.* **11** (2001) 044 [hep-th/0108093].

[49] B. Ponsot, *Liouville theory on the pseudosphere: bulk-boundary structure constant*, *Phys. Lett.* **B 588** (2004) 105 [hep-th/0309211].

[50] E.J. Martinec, G.W. Moore and N. Seiberg, *Boundary operators in 2D gravity*, *Phys. Lett.* **B 263** (1991) 190.

[51] B.H. Lian and G.J. Zuckerman, *New selection rules and physical states in 2D gravity: conformal gauge*, *Phys. Lett.* **B 254** (1991) 417.

[52] V. Pasquier, *Two-dimensional critical systems labelled by Dynkin diagrams*, *Nucl. Phys.* **B 285** (1987) 162.

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